

Graph Representation for the Solutions of Fermat Equation over the General Linear Group $GL_2(\mathbb{Z}_2)$

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Abstract

In 2023, Hadi et al. investigate the solvability of the Fermat equation $X^k + Y^k = Z^k$ over the general linear group $GL_2(\mathbb{Z}_p)$ for $p = 2$ and $p = 3$. They found that, for $k = 2$ there are 12 solutions in $GL_2(\mathbb{Z}_2)$ while for $k = 3$ there are 3 solutions. In this research, we complete the results in $GL_2(\mathbb{Z}_2)$ for any value of k and then represent the solution set as a graph and investigate the properties.

Keywords: Fermat equation, general linear group, graph.

INTRODUCTION

The Pythagorean equation $x^2 + y^2 = z^2$ is known to have infinitely many integer solutions. These solutions can be constructed from any two integers $m > n$ by considering the identity $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$. Here, we have the triple $(m^2 - n^2, 2mn, m^2 + n^2)$ that forms a solution of the equation.

However, the Pythagorean equation does not only have integer solution. It is also solvable over the ring of matrices of complex numbers. As an example, the triple

$$\left(\begin{pmatrix} -2 & 2i & 0 \\ 0 & -2 & 2i \\ 2i & 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 2i & 1 \\ 1 & 0 & 2i \\ 2i & 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 2i & 1 \\ 1 & -2 & 2i \\ 2i & 1 & -2 \end{pmatrix} \right)$$

forms a solution.

If we go further, in number theory, there is a discussion about Fermat equation, that is the equation in the form $x^k + y^k = z^k$ where $k \in \mathbb{N}$, $k > 2$, and $xyz \neq 0$ (also known as Fermat's Last Theorem). This theorem can be proved for $k = 4$ (see [1]). Wiles [2] said that the solution does not exist for $k > 2$. However, Domiaty [3] found the integral matrix solution for $k = 4$, that is $X =$

$\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$ where a, b, c satisfies $a^2 + b^2 = c^2$. So, the equation $x^4 + y^4 = z^4$ has infinitely many 2×2 integral matrix solutions.

More results related to Fermat equation in context of matrices can be seen in Ribenboim [4]. It is also mentioned in Cao [5] that it is important to give the necessary and sufficient conditions for the existence of matrix solutions. Some were solved by Grytczuk [6] by giving the necessary conditions for the existence of 2×2 integral matrix solutions. Furthermore, Khazanov [7] gave the necessary and sufficient conditions for the solvability in $SL_2(\mathbb{Z}), SL_3(\mathbb{Z}),$ and $GL_3(\mathbb{Z})$. In addition, it was shown that the solution does exist in $SL_2(\mathbb{Z})$ if and only if the power in the equation is neither a multiple of 3 nor 4. The detail discussion is in the introduction part of Cao [5].

Next, Mao [8] discusses the solvability of the equation $A^k + B^k = C^k$ in a class of 2×2 matrices for $k \geq 4$. They showed that the equation has infinitely many solutions in the family of positive semidefinite 2×2 symmetric integral matrices.

Hadi [9 – 11] computed the order of $GL_n(\mathbb{Z}_p)$ and in [12] Hadi et al., make a graph representation for some matrices over integer modulo prime. Some applications of graph to represent an algebraic structure can be seen in [13].

MOTIVATION

In [14], Hadi et al. not only show that the Pythagorean equation $x^2 + y^2 = z^2$ has 12 solutions in $GL_2(\mathbb{Z}_2)$ but also that the Fermat equation $x^k + y^k = z^k$ is solvable in $GL_2(\mathbb{Z}_2)$ for any $k > 2$ as well. Moreover, they also show that the equation $x^3 + y^3 = z^3$ is solvable in $GL_2(\mathbb{Z}_3)$. Thus, it is proper to ask whether there exists a pattern of solutions respect to the power, the order, and the moduli.

There is an easy way to analyze the relation between many objects in mathematics, that is by representing it into a graph. However, there can be many ways to construct a graph to represent a specific problem. In this research, we propose a method to visualize the set of solutions of the equation $x^2 + y^2 = z^2$ in $GL_2(\mathbb{Z}_2)$.

METHODS OF RESEARCH

First, we complete the results in $GL_2(\mathbb{Z}_2)$ for any value of k . Then we define the set of vertices and the set of edges as the material of the graph. After the graph is formed, we create its adjacency matrix and compute the eigenvalues.

RESULTS AND DISCUSSION

The group $GL_2(\mathbb{Z}_2)$ consists of 6 elements. Let us order and label those elements as follows: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, W = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$ and $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Here to obtain W we add the lower row of I to the upper row, to obtain S we add the upper row of W to the lower row, and so on. First, we list the power of each element in $GL_2(\mathbb{Z}_2)$ as follows:

1. $I^j = I$ for any $j \geq 0$.
2. $W^{2j} = I$ and $W^{2j+1} = W$ for any $j \geq 0$.
3. $S^{3j} = I, S^{3j+1} = S,$ and $S^{3j+2} = N$ for any $j \geq 0$.
4. $C^{2j} = I$ and $C^{2j+1} = C$ for any $j \geq 0$.
5. $N^{3j} = I, N^{3j+1} = N,$ and $N^{3j+2} = S$ for any $j \geq 0$.
6. $E^{2j} = I$ and $E^{2j+1} = E$ for any $j \geq 0$.

Next, we examine the addition of elements in $GL_2(\mathbb{Z}_2)$. Clearly, $GL_2(\mathbb{Z}_2)$ is closed only for the following combinations:

$$\begin{array}{ccc} I + S = N & I + N = S & S + N = I \\ W + C = E & W + E = C & C + E = W \end{array}$$

We start with the construction of the representation graph of the solution set for $k = 2$. Clearly, the solutions for $k = 2$ are:

$$\begin{array}{ccc} I^2 + S^2 = N^2 & I^2 + N^2 = S^2 & S^2 + N^2 = I^2 \\ W^2 + S^2 = N^2 & W + N^2 = S^2 & S^2 + N^2 = W^2 \\ C^2 + S^2 = N^2 & C^2 + N^2 = S^2 & S^2 + N^2 = C^2 \\ E^2 + S^2 = N^2 & E^2 + N^2 = S^2 & S^2 + N^2 = E^2 \end{array}$$

Now we define the vertices of the graph as the elements in $GL_2(\mathbb{Z}_2)$ which is involved in the solution and if $X, Y, Z \in GL_2(\mathbb{Z}_2)$ satisfies $X^2 + Y^2 = Z^2$ then we create the edge (X, Y) . The graph is thus as follows:

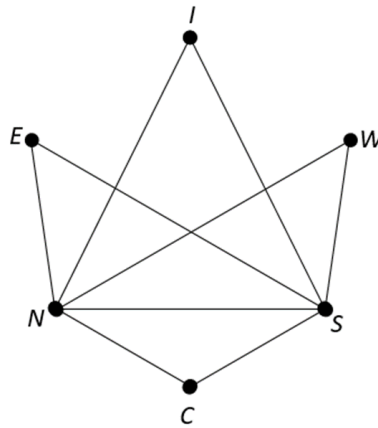


Figure. 1 The representation graph of the solution for $k = 2$

The adjacency matrix of the graph is given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

with the characteristic polynomial $\lambda^6 - 9\lambda^4 - 8$ and eigenvalues $\lambda_1 = \frac{1+\sqrt{33}}{2}$, $\lambda_2 = \frac{1-\sqrt{33}}{2}$, $\lambda_3 = -1$, and $\lambda_4 = \lambda_5 = \lambda_6 = 0$.

Next, we continue with $k = 3$. Clearly, for this case we only have three solutions: $W^3 + C^3 = E^3$, $W^3 + E^3 = C^3$, and $C^3 + E^3 = W^3$. Thus, the graph is exactly K_3 .

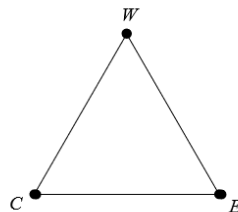


Figure. 2 The representation graph of the solution for $k = 3$

This graph has adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and characteristic polynomial $\lambda^3 - 3\lambda - 2$, and eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = -1$.

Next, it can easily be seen that for $k = 4$ the solution is just the same (we only need to change the power) with those for $k = 2$. So, we continue to the next case, $k = 5$. Notice that $X^5 = X$ for $X \in \{I, W, C, E\}$, $S^5 = N$, and $N^5 = S$. Hence, we have that the solutions are:

$$\begin{array}{lll} I^5 + S^5 = N^5 & I^5 + N^5 = S^5 & S^5 + N^5 = I^5 \\ W^5 + C^5 = E^5 & W^5 + E^5 = C^5 & C^5 + E^5 = W^5 \end{array}$$

Thus, the graph is as follows:

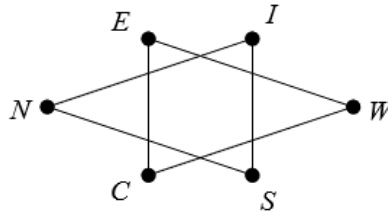


Figure. 3 The representation graph of the solution for $k = 5$

The adjacency matrix of the graph is given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

with the characteristic polynomial $\lambda^6 - 6\lambda^4 - 4\lambda^3 + 9\lambda^2 + 12\lambda + 4$ and eigenvalues $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = -1$.

Now, since $X^6 = I$ for all $X \in GL_2(\mathbb{Z}_2)$ then there is no solution for $k = 6$. Next, we have $X^7 = X$ for all $X \in GL_2(\mathbb{Z}_2)$. So, the solution for $k = 7$ is just the same with those for $k = 5$. We are ready for the conclusion.

CONCLUSION

Let us denote the representation graph for the solution of Fermat equation of $x^k + y^k = z^k$ in $GL_n(\mathbb{Z}_p)$ by $\Gamma(k, n, p)$. Based on the discussion above, we have the following results; (1) if $k \bmod 6 = 2$ or $k \bmod 6 = 4$ then the graph $\Gamma(k, 2, 2)$ is as in Figure 1. This graph has diameter 2 and its complement relative to the complete graph K_6 is the complete graph K_4 and two isolated vertices. The energy of this graph is $|\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| + |\lambda_5| + |\lambda_6| = \left| \frac{1+\sqrt{33}}{2} \right| + \left| \frac{1-\sqrt{33}}{2} \right| + |-1| + |0| + |0| + |0| = \frac{1+\sqrt{33}}{2} + \frac{\sqrt{33}-1}{2} + 1 + 0 + 0 + 0 = 1 + \sqrt{33}$. Next, (2) if $k \bmod 6 = 3$ then the graph $\Gamma(k, 2, 2)$ is exactly K_3 . This graph has diameter 1 and the energy of this graph is $|\lambda_1| + |\lambda_2| + |\lambda_3| = |2| + |-1| + |-1| = 2 + 1 + 1 = 4$. Furthermore, (3) if $k \bmod 6 = 1$ or $k \bmod 6 = 5$ then the graph

$\Gamma(k, 2, 2)$ is as in Figure 3. This graph is not connected and the energy of this graph is $|\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| + |\lambda_5| + |\lambda_6| = |2| + |2| + |-1| + |-1| + |-1| + |-1| = 2 + 2 + 1 + 1 + 1 + 1 = 8$. Lastly, (4) if k is multiple of 6 then there is no solution. We suggest the reader to investigate the graph $\Gamma(k, 2, 3)$ and $\Gamma(k, 3, 2)$ as the continuation of this research.

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