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The Coefficient of the Generating Function in the Form of a Lower Hessenberg Matrix

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Abstract

This study aims to obtain the generating function with the lower Hessenberg matrix coefficients. By using the properties of the lower Hessenberg matrix, then the construction of generating function of the matrix is formed . The result of this research is the coeefficient of generating function which consists of a complex number of power series which is a collection of several terms.

Keywords: properties, power series, term.

Abstrak

Penelitian ini bertujuan untuk mendapatkan fungsi pembangkit dengan koefisien matriks Hessenberg bawah. Dengan menggunakan sifat yang dimiliki oleh dari matriks Hessenberg bawah, maka konstruksi dari pembentukan fungsi pembangkit dari matrix tersebut dapat terbentuk. Hasil dari penelitian ini adalah fungsi pembangkit yang di dalamnya terdiri dari deret pangkat bilangan kompleks yang merupakan kumpulan dari beberapa suku.

Kata-kata kunci: sifat, deret pangkat, suku.

INTRODUCTION

The generating function is a model expressed in the terms of an infinite sequence, which becomes the coefficients of a power series. The generating function can be used to solve problems such as solving recurrence problems, proving combinatorial identities, or various other applications (Wahyuni, 2013).

A lower Hessenberg matrix is defined as a matrix whose entries above the first superdiagonal vanish. In everyday life, matrices play an important role, such as simplifying analysis of complex problems involving multiple factors and being used to solve investigation problems (Khasanah & Kuntarini, 2020).

In the research of Leerawat & Daowsud (2023), several correlation between the determinants of lower Hessenberg matrices, where each entry is composed of terms from a certain sequence, and the generating function of that sequence were discussed. The result obtained was the determinant of several matrices. Based on that research, the researcher intends to prove the generating function with coefficients using a lower Hessenberg matrix.

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METHOD

Before focuss on the main result, lets brief the following theories as the fundamental need to be understood first. Here is some theories from the previous study that beneficial for the next discussion, such as:

Definition 1. (Sari, 2015) Let D is set of point on Z plane. The complex function f is the rule that pairs each point z in D with one and only one point w in the plane W, called (z, w). The function is written as $w = f(z)$.

The set of D is known as the domain of the function f, denoted as D_f and $f(z)$ is referred to as the the value of z by f. Then, the range or image of f is denoted as R_f, which is the set of $f(z)$ for each z in D.

If $z = x + iy$, then the function $w = f(z)$ can be decomposed into $w = u(x, y) + iv(x, y)$ which means that $Re(w)$ and $Im(w)$ are functions defined in terms of the two real variables x and y. If we have the formula of $z = r(\cos\theta + i\sin\theta)$, then we can write $w = u(r, \theta) + iv(r, \theta)$.

Example 1. Express the function $f(z) = 2z^2 - i$ in the form of u and v.
Solution. Let $z = x + iy$,
then the function

$$\begin{aligned} w = f(z) &= 2z^2 - i \\ &= 2(x + iy)^2 - i \\ &= 2(x^2 + 2xyi - y^2) - i \\ &= 2(x^2 - y^2) + i(2xy - 1) \end{aligned}$$

Therefore, we have $u = 2(x^2 - y^2)$ and $v = 2xy - 1$.

Definition 2. (Zetriuslita, 2014) A complex sequence is a sequence of complex numbers arranged in a certain pattern. It is usually written in the form $Z_1, Z_2, Z_3, \cdots$ or $\{Z_1, Z_2, Z_3, \cdots\}$ or simply $\{Z_n\} \cdots$. The numbers $Z_1, Z_2, Z_3, \cdots$ are the sequences with that forms. The term of the sequence Z_n is referred to as the general form of or the n −th form of the sequence.

Example 2. Let the sequence with the following is the form of complex sequence with n term.

$$\left\{\frac{3i}{2^n}\right\} = \frac{3i}{2}, \frac{3i}{4}, \frac{3i}{8}, \dots$$

Definition 3. (Buhaerah, 2019) Let $a_k = a_0, a_1, a_2, \dots, a_n$ be a sequence of numbers. The ordinary generating function of the sequence $\{\alpha_i\}_{i=0}^{\infty}$ is a power series $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

Example 3. Let $a_k = 1$ for $k = 0,1,2, \cdots$, then $G(x) = 1 + x + x^2 + \cdots$ and where the form of $G(x) = \frac{1}{1-x}$, with the condition that must be fulfilled such $|x| < 1$.

Definition 4. (Kaygizis & Sahin, 2013) An $n \times n$ matrix, $H_n = [H_{ij}]$ is called a lower Hessenberg matrix if $h_{ij} = 0$ for $j - i > 1$, that is

$$H_{n\times n} = \begin{bmatrix} h_{11} & h_{12} & 0 & \dots & 0 \\ h_{21} & h_{22} & h_{23} & \cdots & 0 \\ h_{31} & h_{32} & h_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1,1} & h_{n-1,2} & h_{n-1,2} & \cdots & h_{n-1,n} \\ h_{n,1} & h_{n,2} & h_{n,2} & \cdots & h_{n,n} \end{bmatrix}.$$

A Hessenberg matrix (lower Hessenberg matrix) is the matrix with the form of square matrix that closely resembles a matrix with the form of triangular matrix. The difference is that in a triangular (lower) matrix, all entries above the main diagonal are zero, whereas in a Hessenberg matrix, all entries located above the second diagonal (that is, above the main diagonal) are zero.

Example 4. Here is an example of 4 × 4 lower Hessenberg matrix:

$$H_{4\times 4} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 5 & 3 & 6 & 0 \\ 3 & 9 & 7 & 4 \\ 4 & 3 & 8 & 2 \end{bmatrix}.$$

Definition 5. (Syarifudin, et al., 2016) A collection of linear equations in n variables $x_1, x_2, \dots, x_n$ is referred to as a system of linear equations, or simply a linear system. Such a system may have no solution, exactly one solution, or infinitely many solutions.

The linear equations system is a combination of several linear equations with variables $x_1, x_2, \dots, x_n$, such as,

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{matrix}$$

The above of linear equations system has m total equations and n kinds of variables. Next, consider the system of simultaneous linear equations consisting of n equations and m free variables, can be represented in the following form.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Example 5. A linear system consisting of two equations with three variables can be constructed as,

$$\begin{matrix} 4x_1 - x_2 + 3x_3 = -1 \\ 3x_1 + x_2 + 9x_3 = -4 \end{matrix}$$

Definition 6. (Anton, 2000) Let a square matrix called A. The determinant of A, denoted by $det(A)$ or $|A|$ is defined as the sum of all signed elementary products of the entries of A. The resulting value $det(A)$ is referred to as the determinant of A.

Theorem 7. (Anton & Rorres, 2014) If A is an $n \times n$ triangular matrix (whether upper triangular, lower triangular, or diagonal), then the computation of the determinant of A matrix is by operating of the entries on the main diagonal of the matrix, i.e., $|A| = a_{11}a_{22}a_{33} \dots . a_{nn}$.

Theorem 8. (Anton & Rorres, 2014) Let a square matrix called A. If the A matrix has a row or a column of zeros, then $\det(A) = 0$.

Definition 9. (Anton & Rorres, 2014) The determinant of a $n \times n$ size of matrix and called A, then it can be computed by multiplying the entries of any row (or column) with their corresponding cofactors and summing the obtained products, where $1 \leq i \leq n$ dan $1 \leq j \leq n$. The formula

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots + a_{nj}C_{nj}$$

represents the cofactor expansion along the j −th column, while

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in}$$

represents the cofactor expansion along the i −th row.

Example 10. Suppose that

$$M = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}.$$

Then we have $det(M) = (5 \times 3) - (2 \times 4) = 7.$

Definition 11. (Sukiyanto, et al., 2021) Non-negative Integers can be presented as the following kinds,

a. It is known as natural numbers because their symbols start from the positive number 1 and continue, where the set is denoted by enumeration, or can be written as $\{1,2,3,4,5 \ldots\}$.

b. Zero integer

It is usually denoted by the symbol $\{0\}$ which represents an integer that is neither positive nor negative.

It can be concluded that non-negative integers are integers that include zero and positive integers, which consist of the form $\{0,1,2,3,4,5 \ldots\}$.

RESULT AND DISCUSSION

Based on the basic need above understanding, here the main theorem for evaluating the generating function of lower Hessenberg matrix with the detail such the following result.

Theorem 12. (Leerawat & Daowsud, 2023) Let $\{b_n\}_{n\geq 0}$ be an arbitrary sequence of complex numbers, and for each $i = 1,2,\cdots$, let $\{c_{ni}\}_{n\geq 0}$ be a sequence of complex numbers where $c_{0i} \neq 0$. Define the generating functions $B(x) = \sum_{n=0}^{\infty} b_n x^n$ and $C_i(x) = \sum_{n=0}^{\infty} c_{ni} x^n$, for all $i = 1,2,\ldots$, corresponding to the sequences $\{b_n\}_{n\geq 0}$ dan $\{c_{ni}\}_{n\geq 0}$, respectively. Then there exists a generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ such that

$$\sum_{n=0}^{\infty} a_n x^n \, c_{n+1}(x) = B(x)$$

where

$$a_n = \frac{(-1)^n \det(H_{n+1})}{\prod_{i=1}^{n+1} c_{0i}}$$

and with H_{n+1} be the $(n+1) \times (n+1)$ lower Hessenberg matrix that can be defined as

$$H_{n+1} = \begin{bmatrix} b_0 & c_{01} & 0 & 0 & & 0 \\ b_1 & c_{11} & c_{02} & 0 & \cdots & 0 \\ b_2 & c_{21} & c_{12} & c_{03} & & 0 \\ & & \vdots & & \ddots & \vdots \\ b_{n-1} & c_{n-1,1} & c_{n-2,2} & c_{n-3,3} & & c_{0,n} \\ b_n & c_{n,1} & c_{n-1,2} & c_{n-2,3} & \cdots & c_{1,n} \end{bmatrix}$$

for all non-negative integers n.

Proof. For every non-negative integer n, we consider the following system of linear equations

$$\begin{bmatrix} c_{01} & 0 & 0 & & 0 & 0 \\ c_{11} & c_{02} & 0 & \cdots & 0 & 0 \\ c_{21} & c_{12} & c_{03} & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1,1} & c_{n-2,2} & c_{n-3,3} & & c_{0n} & 0 \\ c_{n1} & c_{n-1,2} & c_{n-2,3} & \cdots & c_{1n} & c_{0,n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}.$$

Let,

$$C = \begin{bmatrix} c_{01} & 0 & 0 & & 0 & 0 \\ c_{11} & c_{02} & 0 & \cdots & 0 & 0 \\ c_{21} & c_{12} & c_{03} & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1,1} & c_{n-2,2} & c_{n-3,3} & \cdots & c_{0n} & 0 \\ c_{n1} & c_{n-1,2} & c_{n-2,3} & & c_{1n} & c_{0,n+1} \end{bmatrix}, \quad A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}.$$

Since $det\, C \neq 0$, then the matrix c has inverse or invertible. It implies that the above equation has the unique solution in the following form.

$$A = C^{-1}B$$

For the inverse of the matrix, we have the formula,

$$C^{-1} = \frac{1}{det\, C} adj\ C \tag{1}$$

By substitution process with C^{-1} to the (1) and since

$$adj\ C = \begin{bmatrix} c_{01} & c_{11} & c_{21} & & c_{n-1,1} & c_{n1} \\ 0 & c_{02} & c_{12} & \cdots & c_{n-2,2} & c_{n-1,2} \\ 0 & 0 & c_{03} & & c_{n-3,3} & c_{n-2,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{0n} & c_{1n} \\ 0 & 0 & 0 & & 0 & c_{0,n+1} \end{bmatrix}$$

then we have,

$$\begin{aligned} A &= C^{-1}B \\ &= \left[\frac{1}{det\, C} adj\ C\right] B \\ &= \frac{1}{det\, C} \begin{bmatrix} c_{01} & c_{11} & c_{21} & & c_{n-1,1} & c_{n1} \\ 0 & c_{02} & c_{12} & \cdots & c_{n-2,2} & c_{n-1,2} \\ 0 & 0 & c_{03} & & c_{n-3,3} & c_{n-2,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{0n} & c_{1n} \\ 0 & 0 & 0 & & 0 & c_{0,n+1} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} \\ &= \frac{1}{det\, C} \begin{bmatrix} c_{01}b_0 + c_{11}b_1 + c_{21}b_2 + \cdots + c_{n-1,1}b_{n-1} + c_{n1}b_n \\ 0b_0 + c_{02}b_1 + c_{12}b_2 + \cdots + c_{n-2,2}b_{n-1} + c_{n-1,2}b_n \\ 0b_0 + 0b_1 + c_{03}b_2 + \cdots + c_{n-3,3}b_{n-1} + c_{n-2,3}b_n \\ \vdots \\ 0b_0 + 0b_1 + 0b_2 + \cdots + c_{0n}b_{n-1} + c_{1n}b_n \\ 0b_0 + 0b_1 + 0b_2 + \cdots + 0b_{n-1} + c_{0,n+1}b_n \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \dfrac{c_{01}b_0 + c_{11}b_1 + c_{21}b_2 + \cdots + c_{n-1,1}b_{n-1} + c_{n1}b_n}{\det C} \\ \dfrac{0b_0 + c_{02}b_1 + c_{12}b_2 + \cdots + c_{n-2,2}b_{n-1} + c_{n-1,2}b_n}{\det C} \\ \dfrac{0b_0 + 0b_1 + c_{03}b_2 + \cdots + c_{n-3,3}b_{n-1} + c_{n-2,3}b_n}{\det C} \\ \vdots \\ \dfrac{0b_0 + 0b_1 + 0b_2 + \cdots + c_{0n}b_{n-1} + c_{1n}b_n}{\det C} \\ \dfrac{0b_0 + 0b_1 + 0b_2 + \cdots + 0b_{n-1} + c_{0,n+1}b_n}{\det C} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{c_{01}b_0 + c_{11}b_1 + \ c_{21}b_2 + \cdots + c_{n-1,1}b_{n-1} + c_{n1}b_n}{\det C} \\ \dfrac{0 + c_{02}b_1 + c_{12}b_2 + \cdots + c_{n-2,2}b_{n-1} + c_{n-1,2}b_n}{\det C} \\ \dfrac{0 + 0 + c_{03}b_2 + \cdots + c_{n-3,3}b_{n-1} + c_{n-2,3}b_n}{\det C} \\ \vdots \\ \dfrac{0 + 0 + 0 + \cdots + c_{0n}b_{n-1} + c_{1n}b_n}{\det C} \\ \dfrac{0 + 0 + 0 + \cdots + 0 + c_{0,n+1}b_n}{\det C} \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} \frac{c_{01}b_0+c_{11}b_1+\ c_{21}b_2+\cdots+c_{n-1,1}b_{n-1}+c_{n1}b_n}{\det C} \\ \frac{c_{02}b_1+c_{12}b_2+\cdots+c_{n-2,2}b_{n-1}+c_{n-1,2}b_n}{\det c_{0i}} \\ \frac{c_{03}b_2+\cdots+c_{n-3,3}b_{n-1}+c_{n-2,3}b_n}{\det C} \\ \vdots \\ \frac{c_{0n}b_{n-1}+c_{1n}b_n}{\det C} \\ \frac{c_{0,n+1}b_n}{\det C} \end{bmatrix} \tag{2}$$

Now, let

$$C_{01} = \begin{bmatrix} b_0 & 0 & 0 & & 0 & 0 \\ b_1 & c_{02} & 0 & \cdots & 0 & 0 \\ b_2 & c_{12} & c_{03} & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1} & c_{n-2,2} & c_{n-3,3} & \cdots & c_{0n} & 0 \\ b_n & c_{n-1,2} & c_{n-2,3} & & c_{1n} & c_{0,n+1} \end{bmatrix},$$

then we have the form of $det\, C_{01} = c_{01}b_0 + c_{11}b_1 + \ c_{21}b_2 + \cdots + c_{n-1,1}b_{n-1} + c_{n1}b_n$ represents as the cofactor expansion along the first column and

$$C_{02} = \begin{bmatrix} c_{01} & b_0 & 0 & & 0 & 0 \\ c_{11} & b_1 & 0 & \cdots & 0 & 0 \\ c_{21} & b_2 & c_{03} & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1,1} & b_{n-1} & c_{n-3,3} & \cdots & c_{0n} & 0 \\ c_{n1} & b_n & c_{n-2,3} & & c_{1n} & c_{0,n+1} \end{bmatrix},$$

where $det\, C_{02} = c_{02}b_1 + c_{12}b_2 + \cdots + c_{n-2,2}b_{n-1} + c_{n-1,2}b_n$, obtained from the cofactor expansion along the second column, and the process continues until it reaches the form of

$$C_{0,n} = \begin{bmatrix} c_{01} & 0 & 0 & & 0 & b_0 \\ c_{11} & c_{02} & 0 & \cdots & 0 & b_1 \\ c_{21} & c_{12} & c_{03} & & 0 & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1,1} & c_{n-2,2} & c_{n-3,3} & \cdots & c_{0n} & b_{n-1} \\ c_{n1} & c_{n-1,2} & c_{n-2,3} & & c_{1n} & b_n \end{bmatrix},$$

where $det\, C_{0,n} = c_{0,n+1}b_n$ as the cofactor expansion along the $(n+1) -$ column.

In the next process in, substitute $det\, C_{01}, det\, C_{02}, \cdots,\ det\, C_{0,n}$ to (2), and results is in the following form

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} \dfrac{\det C_{01}}{\det C} \\ \dfrac{\det C_{02}}{\det C} \\ \dfrac{\det C_{03}}{\det C} \\ \vdots \\ \dfrac{\det C_{0,n-1}}{\det C} \\ \dfrac{\det C_{0,n}}{\det C} \end{bmatrix}$$

or can be written as

$$a_0 = \frac{\det C_{01}}{\det C},\quad a_1 = \frac{\det C_{02}}{\det C},\quad a_2 = \frac{\det C_{03}}{\det C}, \dots ,\ a_{n-1} = \frac{\det C_{0,n-1}}{\det C},\quad a_n = \frac{\det C_{0,n}}{\det C}$$

Observe that the determinant of the first factor on the left side of equation (1) is given by $\prod_{i=1}^{n+1} C \neq 0$. Consequently,

$$a_n = \frac{\det C_{0,n}}{\det C}$$

$$a_n = \frac{\begin{bmatrix} c_{01} & 0 & 0 & & 0 & b_0 \\ c_{11} & c_{02} & 0 & \cdots & 0 & b_1 \\ c_{21} & c_{12} & c_{03} & & 0 & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1,1} & c_{n-2,2} & c_{n-3,3} & \cdots & c_{0,n} & b_{n-1} \\ c_{n1} & c_{n-1,2} & c_{n-2,3} & & c_{1,n} & b_n \end{bmatrix}}{\prod_{i=1}^{n+1} c_{0,i}}$$

By applying elementary column operations—specifically, by swapping the first column with the $i-$ column for each $i = 2, 3, 4, \dots, n+1$, we obtain

$$a_n = \frac{(-1)^n \begin{vmatrix} b_0 & c_{01} & 0 & 0 & \dots & 0 \\ b_1 & c_{11} & c_{02} & 0 & \dots & 0 \\ b_2 & c_{21} & c_{12} & c_{03} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & c_{n-1,1} & c_{n-2,2} & c_{n-3,3} & \dots & c_{0,n} \\ b_n & c_{n1} & c_{n-1,2} & c_{n-2,3} & \dots & c_{1,n} \end{vmatrix}}{\prod_{i=1}^{n+1} c_{0,i}}$$

$$= \frac{(-1)^n \det\left(H_{n+1}\right)}{\prod_{i=1}^{n+1} c_{0,i}}$$

By examining an infinite system of linear equations of the form

$$\begin{bmatrix} c_{01} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ c_{11}x & c_{02}x & 0 & \cdots & 0 & 0 & \cdots \\ c_{21}x^2 & c_{12}x^2 & c_{03}x^2 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \\ c_{n-1,1}x^{n-1} & c_{n-2,2}x^{n-1} & c_{n-3,3}x^{n-1} & \cdots & c_{0n}x^{n-1} & 0 & \cdots \\ c_{n1}x^n & c_{n-1,2}x^n & c_{n-2,3}x^n & \cdots & c_{1n}x^n & c_{1,n+1}x^n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1x \\ b_2x^2 \\ \vdots \\ b_{n-1}x^{n-1} \\ b_nx^n \end{bmatrix}$$

then,

$$c_{01}a_0 = b_0$$

$$c_{11}a_0x + c_{02}a_1x = b_1x$$

$$c_{21}a_0x^2 + c_{12}a_1x^2 + c_{03}a_3x^2 = b_2x^2$$

$$\vdots \quad \vdots$$

$$c_{n1}a_0x^n + c_{n-1,2}a_1x^n + c_{n-2,3}a_3x^n + \cdots + c_{1n}a_nx^n = b_nx^n$$

By using the addition process for to the both side of the equation, we have

$$a_0C_1(x) + a_1xC_2(x) + \cdots + a_nx^nC_{n+1}(x) + \cdots = B(x),$$

as is expected.

The result of this proofing presents a coefficient of generating function of lower Hessenberg matrix consist of complex number of the construction of the correlation of $A(x)$ and $C(x)$ performing the $B(x)$. This finding is only applied on lower Hessenberg matrix formation with no longer other kinds of matrices.

CONCLUSION

Let $B(x)$ and $C_i(x)$ be the generating functions of the sequences of complex number of $\{b_n\}_{n\geq 0}$ and $\{c_{ni}\}_{n\geq 0}$ with $\det C \neq 0$, respectively. Then tere is a generating function of $A(x) = \sum_{n=0}^{\infty} a_nx^n$ as the relation of both where

$$\sum_{n=0}^{\infty} a_nx^n\, c_{n+1}(x) = B(x)$$

with a_n is the form of lower Hessenberg output with $a_n = \frac{(-1)^n \det(H_{n+1})}{\prod_{i=1}^{n+1} c_{0,i}}$ construction.

REFERENCES

Anton, H., 2000. Elementary Linear Algebra. 8th penyunt. New York: John Wiley & Sons.

Anton, H. & Rorres, C., 2014. Elementary Linear Algebra. 11th penyunt. USA: Wiley.

Buhaerah, 2019. Matematika Diskrit. 1st penyunt. Parepare: IAIN Parepare Nusantara Press.

Kaygizis, K. & Sahin, A., 2013. Determinants and permanents of Hessenberg matrices and generalized Lucas polynomials. Buletin of The Iranian Mathematical Society, 6(6), pp. 1065-1078.

Khasanah, N. & Kuntarini, A. A. W., 2020. The rule of hessenberg matrix for computing determinant of centrosymmetric matrices. CAUCHY: Jurnal Matematika Murni dan Aplikasi , 6(2), pp. 140-148.

Khasanah, N., Surarso, B. & Farikhin, 2020. Necessary and sufficient on the computation of determinant of a kind of special matrix. AIP Conference Proceedings, Volume 2234, p. 040014.

Leerawat, U. & Daowsud, K., 2023. Determinants of Some Hessenberg Matrices with Generating Functions. Special Matrices 2023; 11: 1 – 8., 11(1), pp. 1-8.

Sari, D. I., 2015. Buku Diktat: Analisa Variabel Kompleks. 1st penyunt. Bangkalan: STKIP PGRI Bangkalan.

Sukiyanto, et al., 2021. Matematika untuk PGSD/ PGMI. 1st penyunt. Yogyakarta: Nuta Media.

Syarifudin, Mikrayanti & Muslim, 2016. Aljabar Linear. 1st penyunt. Bima: LPP Mandala.

Wahyuni, A., 2013. Matematika Diskrit. 1st penyunt. Makassar: Alauddin University Press.

Zetriuslita, 2014. Mudah Memahami Analisis Kompleks. 1st penyunt. Yogyakarta: Fahma Media.