ON THE FORMULATION OF GUAGE TRANSFORMATION FOR THE GROSS-PITAEVSKII EQUATION

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Abstrak

Pada makalah ini telah dirumuskan tranformasi-transformasi gauge untuk persamaan Gross-Pitaevskii tiga dimensi. Metode yang digunakan adalah menuliskan terlebih dahulu bentuk persamaan Gross-Pitaevskii umum dengan memperkenalkan dua fungsi baru yang merupakan medan vektor. Tujuan utama dari makalah ini adalah mendapatkan bentuk eksplisit dari setiap fungsi yang bersangkutan sehingga persamaan Gross-Pitaevskii awalnya dapat dihasilkan. Dalam hal ini telah diterapkan syarat batas bahwa semua panjang vektor yang didapat haruslah real positif meskipun semua fungsi-fungsi yang bersangkutan diperbolehkan kompleks. Hasil perhitungan yang diperoleh menunjukkan bahwa hanya terdapat dua kemungkinan transformasi yang diperbolehkan jika syarat batas tersebut diterapkan.

Abstract

We have formulated the gauge transformations of the three-dimensional Gross-Pitaevskii equation. The applied method is to initially write the general Gross-Pitaevskii equation by introducing two functions constituted as vector fields. The main purpose of this paper is to obtain the explicit forms for every appropriate function in order to produce the present Gross-Pitaevskii equation. In this case, we have imposed the condition that all the lengths of each vector must be positive real even though we allow all the complex functions. The achieved results show two possibilities of the suitable transformations if the boundary condition is fixed.

Keywords
Gauge transformation, Gross-Pitaevskii, Bose-Einstein condensation

1. Introduction

It has been well-known that the concept of gauge invariance has been introduced a long time ago by Hermann Weyl by considering the relation between the connection in general relativity and the potentials in electromagnetic [1, 2]. Later, in the middle of 20th century the gauge invariance has entered the area of quantum mechanics when the local phase transformation was considered. At that time one found that the gauge transformation in electromagnetic must be applied to keep the Schrödinger equation to be gauge invariance after the wave function had been transformed through the local phase transformation. Finally, the Schrödinger equation had to be modified by including both the scalar and vector potentials in the electromagnetic. That was the birth of the interaction between the integer particles called boson and the half-integer particles called fermion [2].

In the other case, the existence of Bose gases has been predicted by S. N. Bose and developed later by Einstein in that century, and observed in 1995 by several experiments where the alkali atoms were confined in the magnetic trap and cooled down in the $T \cong 0$ K [3]. The appropriate successful experiments can be seen in [4, 5] and the theoretical considerations described by the Gross-Pitaevskii equation (GPE) can also be found in [6-10], for the detailed reviews interested readers are welcome to see [11, 12]. Surprisingly, the development of the Bose gases applications have been widely explored in some cases, for example the atom laser [13-15].

In fact, some interesting cases of the relation between GPE and gauge transformations can be found in [16-18]. In this paper, we explore another discussion of the gauge transformation by following the proposed method of Ricardo-Soto *et al.* when they discussed some models of the one-dimensional quantum oscillators [19]. In their paper, they initially proposed the new Hamiltonian which generally was not assumed self-adjoint. Since the GPE can be built in the gauge invariance form, we motivate to explore all the possibilities of gauge transformations by following the method.

We organize the rest of paper as follows. We formulate the modified GPE which is invariant under the appropriate transformations after applying the local phase transformation in sec. 2. Next, in sec.3 we report our results based on our calculations and the imposed condition. We will be giving our conclusions in sec.4.

2. Methods

Firstly, we consider the three-dimensional GPE which can be written in the Cartesian coordinate [7, 9]

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + U|\Psi|^2\Psi + V(\vec{r})\Psi\,, \tag{1}$$

where $\Psi = \Psi(\vec{r}, t)$ represents the wave function of condensate, $V(\vec{r})$ denotes the usual anisotropic parabolic trapping potential

$$V(\vec{r}) = \frac{m\omega^2}{2}({\lambda_1}^2x^2 + {\lambda_2}^2y^2 + {\lambda_3}^2z^2)\,, \tag{2}$$

and $U = 4\pi\hbar^2 a/m$ describes the internal interaction in the condensate.

In order to obtain the appropriate functions via gauge transformations which produce Eq. (1), we initially propose the modified GPE by pursuing Cordero-Soto *et al.* [19]

$$i\hbar\left(\frac{\partial}{\partial t} + i\phi\right)\psi = -\frac{\hbar^2}{2m}\left(\nabla - i\vec{A}\right)^2\psi + \left(\nabla - i\vec{A}\right)\bullet\vec{C}\psi + \vec{W}\bullet\left(\nabla - i\vec{A}\right)\psi$$
$$+U|\psi|^2\psi = 0\,. \tag{3}$$

Next we consider the following similar equation

$$i\hbar\left(\frac{\partial}{\partial t} + i\tilde{\phi}\right)\tilde{\psi} = -\frac{\hbar^2}{2m}\left(\nabla - i\tilde{\vec{A}}\right)^2\tilde{\psi} + \left(\nabla - i\tilde{\vec{A}}\right)\bullet\tilde{\vec{C}}\tilde{\psi}$$
$$+\tilde{\vec{W}}\bullet\left(\nabla - i\tilde{\vec{A}}\right)\tilde{\psi} + U|\tilde{\psi}|^2\tilde{\psi} = 0\,, \tag{4}$$

which is invariant via the transformations below

$$\tilde{\vec{A}} = \vec{A} + \nabla\chi\,, \tag{5}$$

$$\tilde{\phi} = \phi - \frac{\partial\chi}{\partial t}\,, \tag{6}$$

$$\tilde{\vec{C}} = \vec{C}\,, \tag{7}$$

$$\tilde{\vec{W}} = \vec{W}\,, \tag{8}$$

if the local phase transformation of the wave function

$$\tilde{\Psi} = e^{i\chi}\Psi\,, \tag{9}$$

is applied

Here, we have to find the explicit forms of the functions $\vec{A}$, ϕ, χ, $\vec{C}$, and $\vec{W}$ without limiting their values.

3. Results and Discussion

After we employ the imposed condition that all the lengths of each vector must be real we find only two possibilities. These possibilities occur since we have to set two functions, namely $\vec{A}$ and χ. In this case $\vec{A}$ must be set zero while χ should be an arbitrary function of time, $\chi = f(t)$.

Possibility 1

$$\phi = 0\,, \tag{10}$$

$$\vec{C} = \frac{m\omega^2}{6}\left(\lambda_1{}^2 x^3\hat{i} + \lambda_2{}^2 y^3\hat{j} + \lambda_3{}^2 z^3\hat{k}\right), \tag{11}$$

$$\vec{W} = -\frac{m\omega^2}{6}\left(\lambda_1{}^2 x^3\hat{i} + \lambda_2{}^2 y^3\hat{j} + \lambda_3{}^2 z^3\hat{k}\right). \tag{12}$$

thus

$$\tilde{\tilde{\vec{A}}} = 0, \tag{13}$$

$$\tilde{\phi} = -\frac{df}{dt}, \tag{14}$$

$$\tilde{\tilde{\vec{C}}} = \frac{m\omega^2}{6}\left(\lambda_1{}^2 x^3\hat{i} + \lambda_2{}^2 y^3\hat{j} + \lambda_3{}^2 z^3\hat{k}\right), \tag{15}$$

$$\tilde{\tilde{\vec{W}}} = -\frac{m\omega^2}{6}\left(\lambda_1{}^2 x^3\hat{i} + \lambda_2{}^2 y^3\hat{j} + \lambda_3{}^2 z^3\hat{k}\right). \tag{16}$$

Possibility 2

The employed condition in this case is to set $\vec{C} = \vec{b}(t)$ as an arbitrary function of time. Then we obtain

$$\phi = \frac{m\omega^2}{2\hbar}\left(\lambda_1{}^2 x^2 + \lambda_2{}^2 y^2 + \lambda_3{}^2 z^2\right), \tag{17}$$

$$\vec{W} = -\vec{b}(t) \quad . \tag{18}$$

Therefore we can easily write

$$\tilde{\tilde{\vec{A}}} = 0, \tag{19}$$

$$\tilde{\phi} = \frac{m\omega^2}{2\hbar}\left(\lambda_1{}^2 x^2 + \lambda_2{}^2 y^2 + \lambda_3{}^2 z^2\right) - \frac{df}{dt}, \tag{20}$$

$$\tilde{\tilde{\vec{C}}} = -\tilde{\tilde{\vec{W}}} = \vec{b}(t). \tag{21}$$

4. Conclusion

We have written two possibilities in the construction of gauge transformations to keep the current GPE. In this case, we have absolutely set $\vec{A} = 0$ and $\chi = f(t)$ in order to obtain the real-valued of all the lengths of each vector although we enable the complex values for the functions.

Here we also inform that we neglect the meanings of the symbols ϕ and $\vec{A}$ which have been well-known in classical electrodynamics as the scalar and vector potential respectively, for example see [20]. By ignoring the definitions, we are enabled to choose them as the complex-valued functions.

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