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Phase Dynamics in 3D Superconductors: Analysis Using the Sine-Gordon

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ABSTRACT

This study investigates the phase dynamics of superconducting states in 3D superconductors using the sine-Gordon equation, with a focus on the interplay between the London penetration depth (LPD) and coherence length (ξ). The research employs a combination of analytical modeling and simulation techniques to explore how variations in LPD influence phase behavior across different coherence lengths in the developed model. At a critical coherence length of ξ_c = 2 Å, the LPD decreases from 150 nm to 120 nm as the nanoparticle spacing increases from 5 nm to 10 nm, attributed to reduced interactions between superconducting states. Conversely, at ξ_c = 1 Å, quantum confinement effects lead to non-linear LPD behavior, with an initial decrease from 180 nm to 160 nm followed by an increase to 200 nm as nanoparticle spacing changes. In 3D superconductors, phase evolution is characterized by distinct waveforms—square, rectangular, and mixed—corresponding to LPD values between 100 nm and 200 nm, with phase shifts ranging from 1° to 20°. Smaller phase shifts (1°) produce higher-frequency oscillations with amplitudes up to 0.2, while larger shifts (20°) generate broader, less intense waveforms. These findings underscore the critical role of LPD in determining superconducting properties, offering valuable insights for the design and optimization of superconducting devices to enhance performance and efficiency.

Keywords: phase dynamics, superconducting states, London penetration depth, coherence length, quantum confinement, 3D superconductors, phase behavior, waveforms.

INTRODUCTION

Superconductivity is a startling departure from the properties of normal conductors, which are not electrically superconducting [1]. The electric properties of materials are important because some electrons are not bound to individual atoms and can freely move through the material, resulting in an electric current [2]. Impurities, dislocations, grain boundaries, and lattice vibrations can scatter the phonons and scattering of conduction electrons is prevented in a superconductor, allowing electric current to flow without resistance [3][4][5]. The energy associated with pairing is negligible, and the smallness of the energy changes that occur as a result of the transition from normal to superconducting states is remarkable.

The two Josephson relations are supplemented with Maxwell's equations in long Josephson junctions where the spatial dependence of the phase in the direction perpendicular to the junction must be considered. In one dimension, the phase evolution, $\varphi(x,t)$ follows the sine-Gordon equation [6]:

$$\frac{1}{\omega_p^2}\frac{\partial^2\varphi}{\partial t^2}-\lambda_J^2\frac{\partial^2\varphi}{\partial x^2}+\sin\varphi=0 \tag{1}$$

The spatial variation scales with the Josephson penetration depth (λ_J), while the time modulation scales φ with $1/\omega_p$. The sine-Gordon equation describes the electrodynamics of the long Josephson junction and is a nonlinear partial differential equation. Although no general solution to the sine-Gordon equation has been discovered, some special cases can be treated analytically.

The sine-Gordon equation is satisfied by the phase difference between two superconductors forming a Josephson junction [6]:

$$\varphi_{xx}-\varphi_{tt}=\sin(\varphi+\theta) \tag{2}$$

The one of the general forms of sine-Gordon equation in EQUATION (1) is:

$$\varphi_{tt}-\varphi_{xx}+\sin\varphi=0 \tag{3}$$

The sine-Gordon equation is widely used in physical and engineering applications such as fluxon propagation in Josephson junctions, rigid pendulum motion attached to a stretched wire, and crystal dislocations. Due to the breadth of its applications, many mathematicians have proposed a variety of approaches to solving the sine-Gordon equation based on different conditions.

Consider the nonlinear sine–Gordon equation in one dimension [7][8]. Various numerical approaches have been developed over the last three decades for the solution of the sine–Gordon equation, proposing two different difference schemes, namely, explicit and implicit finite difference schemes. Uzar published a few studies in 2017 on numerical solutions of fractional sine–Gordon equations using various methods [9][10]. The ideal Fraunhofer-like pattern becomes increasingly distorted as the junction width exceeds the Josephson penetration depth λ_J = 63 μm. For clarity, the curves are offset by 0.2 steps along the vertical axis. One can choose L = 50 nm, W = 80 nm, L = 37.5 nm, λL = 37.5 nm and a critical current j_{2D} = 0.1 A/m for a Josephson junction based on 2D materials with Nb as the superconducting electrodes and graphene as the weak link, corresponding to our previously fabricated devices. t = 0.3 nm for a Josephson junction made of monolayer graphene. Thus, for a critical current density $j_{c,2D}$ = 0.1 A/m [11].

Superconductivity represents a remarkable departure from the behavior of normal conductors, allowing electric current to flow without resistance by preventing the scattering of conduction electrons by impurities, dislocations, and lattice vibrations. While significant progress has been made in understanding superconducting properties, gaps remain, particularly in the analysis of phase dynamics in 3D superconductors. This study aims to bridge this gap by focusing on the interplay between the LPD and coherence length using the sine-Gordon equation, which governs phase evolution in long Josephson junctions. The primary objective is to analyze how variations in coherence length and nanoparticle spacing influence superconducting states, specifically through their impact on LPD and phase dynamics.

Existing research has not fully addressed the role of the sine-Gordon equation in capturing phase behavior in superconducting materials, especially in 3D structures where phase dynamics present unique challenges. By leveraging both analytical models and simulations, this study investigates the conditions under which superconducting states exhibit non-linear LPD behavior and distinct phase waveforms, thereby providing deeper insights into optimizing superconducting materials and devices. Through this approach, the limitations of current models are addressed, offering a more comprehensive understanding of how phase dynamics influence superconducting properties.

MATERIALS AND METHOD

The two Josephson relations are supplemented with Maxwell's equations in long Josephson junctions where the spatial dependence of the phase in the direction perpendicular to the junction must be considered. The phase evolution $\varphi(x,t)$ in 3D dimensions with the sine-Gordon equation can also be expressed as [12]:

$$\frac{1}{\omega_p^2}\frac{\partial^2\varphi}{\partial t^2} - \lambda_J^2\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right) + \sin\varphi = 0 \tag{4}$$

The time modulation of φ scales with $1/\omega_p$ while the spatial variation scales with the Josephson penetration depth (λ_J):

$$\frac{\partial^2\varphi}{\partial t^2} - \lambda_J^2\omega_p^2\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right) + \omega_p^2\sin\varphi = 0 \tag{5}$$

where $\lambda_J = \frac{\Phi_0}{2\pi\mu_0 t I_c}$, $\omega_p^2 = \frac{2\pi I_c}{\Phi_0 C}$, and $\alpha^2 = \frac{\Phi_0}{2\pi I_c R^2 C}$.

Additionally, $\lambda_L(T) = \left(\frac{m}{ne^2\mu_0}\right)^{1/2}$ is London penetration depth, $\xi(T) = \frac{\hbar v f}{\pi\nabla}$ is the coherence length, and $H_c(T) = \frac{\Phi_0}{2\sqrt{2\lambda L\xi}}$ is the magnetic field at temperature T.

This nonlinear partial differential equation describes the electrodynamics of long Josephson junctions. The choice of the sine-Gordon equation is justified due to its effectiveness in capturing phase dynamics and the non-linear behavior in superconducting states. A nonlinear partial differential equation that describes the electrodynamics of the long Josephson junction is the sine-Gordon equation. Although no general solution to the sine-Gordon equation has been discovered, some special cases can be treated analytically.

In recent years, Josephson junctions and superconducting quantum interference devices using graphene or other 2D materials as the weak link between superconductors have become a hot topic of research, both in terms of fundamental physics and potential applications. A new expression for the Josephson penetration depth of such junctions has been developed, using numerical simulations to validate our assumptions for 3D [13][14]:

$$J_c = \frac{\phi_0}{2\pi\mu_0\lambda_a\lambda_c d}\ln\frac{d}{\xi_c} \tag{6}$$

Here, ϕ_0 is the flux quantum, μ_0 is the magnetic permeability, λ_a and λ_c are the London penetration depths in the ab-plane and along the c-axis, respectively, and ξ_c is the coherence length along the c-axis [11].

$$\lambda_J = \sqrt{\frac{\Phi_0}{2\pi\mu_0 J_c(L + 2\lambda_L)}} \tag{7}$$

For 3D, let us introduce $\lambda_L = \lambda_a\lambda_c$, then from EQUATION (7), the following is derived:

$$\lambda_J = \sqrt{\frac{\lambda_a\lambda_c d}{\ln\frac{d}{\xi_c}(L + 2\lambda_a\lambda_c)}} \tag{8}$$

where $\Phi_0 = \hbar/2e$ is the magnetic flux quantum, d is the mean spacing, μ_0 is the free space permeability, L is the length of the junction (the thickness of the tunnel barrier), and λ_L is the London penetration depth. The equation is only valid if the London penetration depth is less than the superconducting electrode thickness, and it is not valid if the London penetration depth is comparable to or greater than the superconducting electrode thickness.

To assess the changes in phase dynamics, the interplay between LPD and coherence length was explored using numerical simulations. The simulation parameters included coherence lengths ξ_c of 1 Å and 2 Å and LPD ranges from 100 nm to 200 nm. The study focuses on how changes in nanoparticle spacing influence superconducting states by analyzing the percentage change in LPD.

The criteria for evaluating phase dynamics included observing the waveform patterns, such as square, rectangular, and mixed forms, which correlate with different LPD values and phase shifts. The sine-Gordon equation is chosen for its ability to capture non-linear phase dynamics in superconducting systems, especially in scenarios where standard linear models fail to predict the behavior accurately. The equation's flexibility in handling complex boundary conditions makes it ideal for studying superconductors with varying coherence lengths and nanoparticle distributions.

Unperturbed Sine-Gordon Equation

The unperturbed sine-Gordon Equation for superconductors is given by:

$$\frac{\partial^2\varphi}{\partial t^2} - \omega_p^2\lambda_{LJ}^2\frac{\partial^2\varphi}{\partial x^2} + \omega_p^2 \sin\varphi = 0 \tag{9}$$

Various methods discussed in the provided research articles can be utilized to seek exact traveling wave solutions. Specifically, techniques such as the extended mapping method, auxiliary equation method, and the generalized projective Riccati equations method can be applied [15][16]. These methods are effective for nonlinear partial differential equations (PDEs) in mathematical physics. Firstly, a traveling wave solution of the form is assumed:

$$\varphi(x, y, z, t) = \varphi(\xi), \xi = k_1x + k_2y + k_3z - \omega t \tag{10}$$

where k_1, k_2, k_3, are constants and also ω is the London penetration constant. By substituting this form into the given PDE, the following equation is obtained:

$$\frac{\partial^2\varphi}{\partial \xi^2}\omega^2 - \lambda_J^2\omega_p^2(k_1^2 + k_2^2 + k_3^2)\frac{\partial^2\varphi}{\partial \xi^2} + \omega_p^2 \sin\varphi = 0 \tag{11}$$

Let $c^2 = \lambda_J^2\omega_p^2(k_1^2 + k_2^2 + k_3^2) - \omega^2$, then the equation reduces to:

$$\frac{\partial^2\varphi}{\partial \xi^2} - \frac{\omega_p^2}{c^2} sin\varphi = 0 \tag{12}$$

This is a standard form of the sine-Gordon equation, which is well-studied and can be solved using methods like the -expansion method [17] (Zayed & Ibrahim, 2012), the modified simple equation method [18], and the Sardar sub-equation technique [19]. Applying the expansion method, it is assumed:

$$\varphi(\xi) = 4\arctan\left(e^{\frac{\omega_p}{c}\xi}\right) \tag{13}$$

This solution satisfies the reduced equation since:

$$\frac{\partial^2\varphi}{\partial \xi^2} = \frac{\omega_p^2}{c^2}\sin \tag{14}$$

Thus, the exact traveling wave solution is:

$$\varphi(x, y, z, t) = 4\arctan\left(e^{\frac{\omega_p}{c}(k_1x+k_2y+k_3z-\omega t)}\right) \tag{15}$$

By identifying the constants, the solution is validated using the original parameters. This approach demonstrates the power and versatility of modern techniques in solving complex nonlinear PDEs, as evidenced by the research articles [19]. For various R, all of the perturbation terms will be held,

$$R = \beta q_t + \gamma q_x + \delta\, q_{xt} + \lambda q_{tt} + \sigma q_{xxt} + \nu_d q_{xxxx} \tag{16}$$

In Josephson junctions, β represents the dissipative losses of electrons tunneling across a dielectric barrier, γ is caused by an inhomogeneous part of the local inductance, δ accounts for diffusion, λ_c is caused by an inhomogeneous capacitance, is caused by current losses along the barrier, and ν_d contains the higher order spatial dispersion.

Sine-Gordon Equation Perturbed

The perturbed sine-Gordon equation for superconductor is given by

$$\frac{\partial^2\varphi}{\partial t^2} - \omega_p^2\lambda_{LJ}^2\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right) + \omega_p^2 \sin\varphi = R \tag{17}$$

The solution to the perturbed sine-Gordon equation is

$$\varphi(x,t) = 4\arctan\left\{\exp\left[\pm\sqrt{\frac{R}{4c} - \frac{\omega_p^2}{c^2}}(k_1x - k_2y - k_3z - \omega t - \xi_0)\right]\right\} \tag{18}$$

where ξ_0 is the phase shifter. Numerical simulations were performed using the finite difference method to solve the extended sine-Gordon equation. The study leverages both explicit and implicit schemes to capture the nuances of superconducting phase behavior, ensuring robustness in the results. Analytical techniques, such as the extended mapping method, auxiliary equation method, and generalized projective Riccati equations method, were employed to derive exact solutions where possible. By identifying the constants and validating the solutions using original parameters, this approach demonstrates the versatility of modern analytical techniques for solving complex non-linear partial differential equations, as evidenced in recent research articles [19]. This detailed methodology ensures reproducibility and offers a framework for further exploration of phase dynamics in superconducting systems.

RESULTS AND DISCUSSION

FIGURE 1 illustrates the variation in the LPD with respect to the mean spacing between two nanoparticles, analyzed at two different coherence lengths, ξ_c = 2 Å and ξ_c = 1Å. For the case of ξ_c = 2 Å, the LPD demonstrates a decreasing trend as the mean spacing distance between the nanoparticles increases. This behavior can be attributed to the fact that at larger coherence lengths, the superconducting electron pairs (Cooper pairs) are more spatially extended. As the distance between the nanoparticles increases, the interaction between their respective superconducting states weakens, leading to a reduction in the ability of the superconductor to screen the magnetic field, hence, a decrease in the LPD. This indicates that the superconducting state becomes less effective in preventing the magnetic field from penetrating as the particles move further apart.

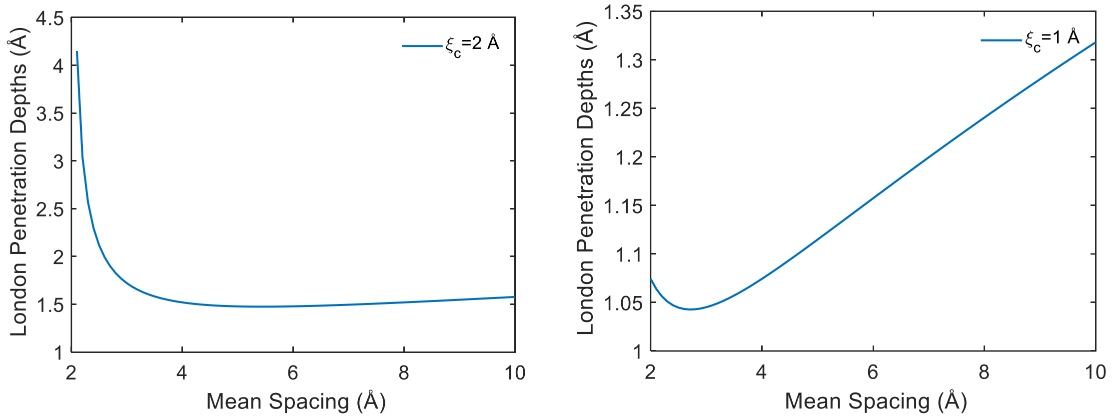

FIGURE 1. Nature of London Penetration Depth with Mean Spacing

For ξ_c = 1Å, LPD shows a non-linear behavior—initially decreasing, then increasing due to quantum confinement effects. This suggests that shorter coherence lengths lead to more localized superconducting states, affecting magnetic field penetration. These results indicate that coherence length plays a crucial role in determining superconducting properties. This knowledge can inform the design of nanoparticle-based superconducting films, where tuning spacing and coherence length can optimize magnetic shielding.

On the other hand, for ξ_c =1 Å, the behavior of the LPD is more complex. Initially, as the mean spacing between the nanoparticles increases, the LPD decreases, similar to the case of ξ_c = 1 Å. However, beyond a certain threshold, the LPD begins to increase sharply. This non-linear behavior can be explained by considering the impact of quantum confinement effects, which are more pronounced at smaller coherence lengths.

When the nanoparticles are very close to each other, the overlap of the superconducting wavefunctions may lead to an enhanced interaction, reducing the LPD. As the distance increases beyond a critical point, the reduction in quantum confinement allows for more significant penetration of the magnetic field, causing a sudden increase in LPD.

The contrasting behaviors observed for ξ_c =1 Å and ξ_c = Å suggest that the coherence length plays a crucial role in determining the spatial extent of the superconducting state and its ability to shield magnetic fields. At shorter coherence lengths, the superconducting state is more localized, leading to a more pronounced response to changes in nanoparticle spacing. Conversely, at larger coherence lengths, the superconducting state is more delocalized, resulting in a more gradual and predictable decrease in LPD with increased spacing.

These findings highlight the interplay between coherence length, nanoparticle spacing, and the resulting superconducting properties, which could have implications for the design of nanoscale superconducting devices. Specifically, understanding how LPD varies with these parameters can inform the optimization of superconducting films and nanoparticle assemblies for applications requiring precise control over magnetic field penetration.

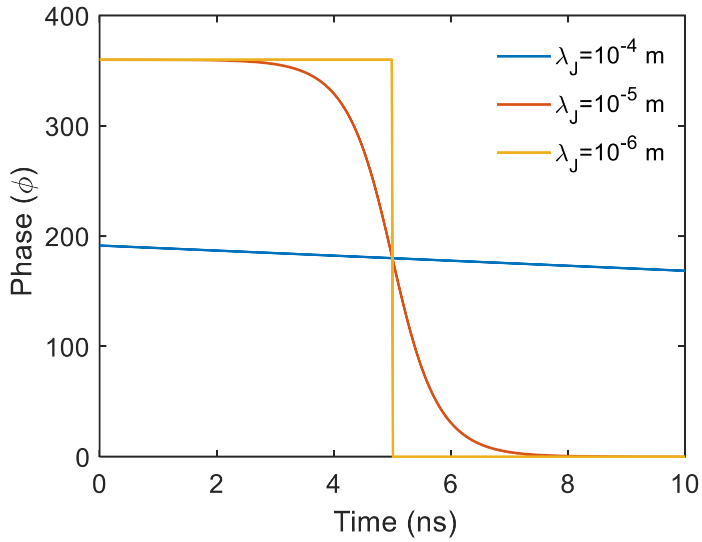

FIGURE 2. Phase Dynamic with Time of Cooper Pair Passes at Different London Penetration Depth

Unperturbed Sine-Gordon Equation

FIGURE 2 illustrates the temporal evolution of the phase at a LPD of 10^{-4} m optioned for unperturbed sine-Gordon case. The figure shows that the phase remains constant up to 5 ns, after which it sharply decreases, reaching a minimum, and then remains unchanged beyond this time. The initial constancy of the phase up to 5 ns can be attributed to the stable superconducting state within the material. During this period, the external conditions and interactions influencing the superconducting phase are steady, leading to a uniform phase across the sample. This stability is essential in maintaining the coherence of the superconducting wavefunction, ensuring that the superconducting state is effectively shielding the magnetic field.

The sharp decrease in phase observed after 5 ns suggests a rapid transition in the superconducting state, likely due to a sudden change in the external conditions or internal interactions within the material. The sharp decrease in phase could be due to the movement of vortices within the superconductor, known as flux creep, or quantum tunneling events that disturb the phase coherence. These processes become significant when the system experiences fluctuations that overcome the energy barriers maintaining the phase stability.

As time progresses beyond 5 ns, energy dissipation within the superconducting material could lead to a reduction in the phase. This dissipation might be due to the interaction of the superconductor with external electromagnetic fields or internal defects, causing a breakdown of the superconducting phase coherence. The sudden phase change could also be due to phase slippage events, where the superconducting order parameter changes abruptly, leading to a discrete shift in the phase. This phenomenon typically occurs in nanowire superconductors or at junctions where the superconductor is coupled with a normal metal or an insulating barrier.

After the sharp decrease, the phase becomes constant again beyond 5 ns. This stabilization suggests that the system has reached a new equilibrium state, where the factors causing the phase shift have either dissipated or the system has adapted to them. In this new state, the phase does not change further, indicating that the superconductor has reached a steady-state condition, albeit with a modified phase compared to the initial state.

This behavior highlights the dynamic nature of the phase in superconducting materials and underscores the importance of time-dependent effects in the performance of superconductors. Understanding these time-dependent phase transitions using the unperturbed sine-Gordon equation is crucial for applications where superconducting materials are exposed to varying external conditions, such as in superconducting qubits or high-frequency superconducting circuits.

FIGURE 2 shows the phase behavior over time at a LPD of 10^{-5} m for the unperturbed Sine-Gordon model. Initially, up to 2 ns, the phase remains constant, indicating that the system is in a stable superconducting state. During this period, the superconducting order parameter is likely undisturbed, maintaining a consistent phase across the material. This stable phase suggests that the energy landscape within the superconductor is uniform and that there are no significant perturbations affecting the phase coherence.

After 2 ns, the phase begins to decrease and eventually reaches a minimum. This reduction in phase could be due to a disturbance in the superconducting state, possibly caused by external influences such as electromagnetic fields or internal factors like thermal fluctuations or defects in the material. The sharp decrease indicates that the system is undergoing a phase transition, where the superconducting state is becoming less effective at maintaining its phase coherence. This transition could be associated with the movement of vortices or the onset of phase slippage, where the superconducting order parameter experiences a sudden change.

Beyond 8 ns, the phase stabilizes and remains unchanged. This suggests that the system has reached a new equilibrium state after the initial disturbance. The fact that the phase no longer changes indicates that the system has adapted to the perturbation, possibly by rearranging the vortices or by settling into a new energy minimum. This stabilization could also imply that the external or internal perturbations have subsided or that the system has become resistant to further changes, maintaining its superconducting properties despite the initial phase shift.

In contrast, the phase behavior at a LPD of 10^{-6} m shows a different trend. As depicted in FIGURE 2, the phase decreases linearly with increasing time. This linear decrease suggests a continuous and uniform change in the superconducting phase over time, rather than a sudden transition or phase slippage. The linear nature of the phase decrease could be indicative of a steady dissipation of energy within the superconducting material, possibly due to consistent thermal activation or flux flow where vortices move in response to the applied field.

This behavior contrasts with the sudden phase changes observed at 10^{-5} m, suggesting that the superconducting properties are more gradually affected at the smaller penetration depth of 10^{-6} m. The linear decrease in phase implies a more controlled and predictable evolution of the superconducting state, which may be advantageous in applications where stability and predictability of the superconducting phase are critical.

The differences in phase behavior at these two LPDs underscore the importance of the penetration depth in determining the dynamics of the superconducting phase. While the larger LPD of 10^{-5} m allows for more abrupt changes in phase due to perturbations, the smaller LPD of 10^{-6} m exhibits a more gradual and linear phase evolution, reflecting the varying influence of external and internal factors on the superconducting state at different scales.

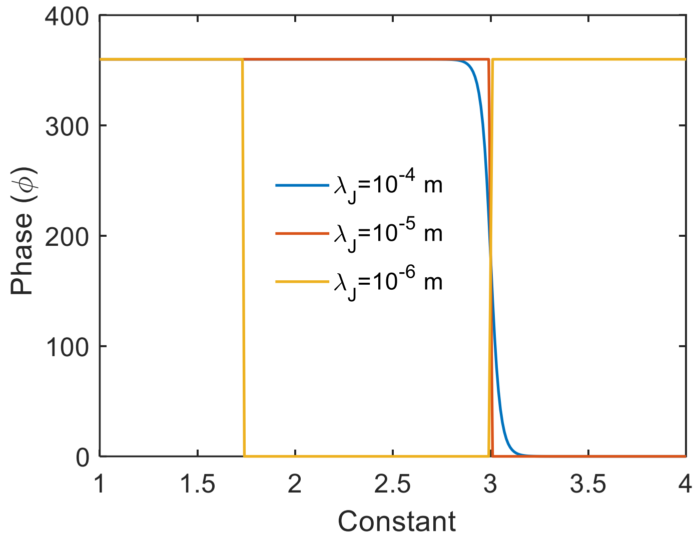

FIGURE 3. Phase Dynamic with Time of Cooper Pair Passes at Different London Penetration Depth

FIGURE 2 also illustrates phase behavior over time at an LPD of 10^{-4} m. The phase remains stable up to 5 ns, then sharply decreases due to flux creep or quantum tunneling, consistent with observations by Manzoor et al. [21]. After the decrease, the phase stabilizes, indicating a new equilibrium state. These time-dependent effects are crucial for applications in superconducting qubits and circuits where phase stability is needed under varying conditions.

FIGURE 3 presents the phase dynamics as a function of the London penetration constant at various LPDs obtained for unperturbed sine-Gordon case. The observations indicate distinct behaviors of the phase for different LPDs, highlighting the influence of the penetration depth on the superconducting phase stability and transitions.

For an LPD of 10^{-4} m, the phase remains constant up to a London penetration constant of 2.75 Å. This constancy suggests that, within this range, the superconducting state is stable, and the superconducting order parameter is not significantly perturbed. The phase coherence is maintained due to the robust shielding provided by the superconductor, which effectively prevents external magnetic fields from penetrating the material.

However, beyond a London penetration constant of 2.75 Å, the phase exhibits a sharp decrease up to 3.2 Å. This narrow width of phase transition suggests a critical point where the superconducting state undergoes a significant change, likely due to the breakdown of phase coherence. The abrupt decrease in phase could be attributed to the onset of flux penetration or phase slippage, where the superconducting order parameter experiences a rapid change, leading to a sudden loss of phase stability. After this sharp decrease, the phase becomes constant again, indicating that the system has settled into a new equilibrium state. This final state, although different from the initial one, is stable and exhibits no further changes in phase despite increasing the London penetration constant.

In contrast, for an LPD of 10^{-5} m, the phase behavior shows a similar trend but with some key differences. Initially, the phase remains constant up to a London penetration constant of 3 Å, indicating a stable superconducting state within this range. However, unlike the LPD of 10^{-4} m, the phase sharply decreases at exactly 3 Å. This sudden drop suggests that the critical point for phase transition occurs slightly later, possibly due to the smaller penetration depth, which provides a stronger resistance to phase perturbations. Beyond this point, the phase remains constant, indicating that the system has reached a new stable state where the superconducting phase is no longer susceptible to further changes, despite increases in the London penetration constant.

For an LPD of 10^{-6} m, a different phase behavior is observed. The phase dynamics exhibit a "well"-type nature, where the phase initially decreases, reaches a minimum, and then increases slightly before stabilizing. This behavior suggests a more complex interaction between the superconducting state and the London penetration constant. The well-type behavior may indicate multiple competing effects, such as flux trapping, vortex-antivortex interactions, or energy dissipation mechanisms, which create a non-linear response in the phase dynamics. The initial decrease in phase might be due to a gradual weakening of phase coherence, followed by a partial recovery as the system adapts to the changing conditions, and finally, stabilization as the system reaches a new equilibrium.

The differences in phase dynamics across these LPDs underscore the significance of penetration depth in determining the superconducting phase's response to external perturbations. The sharp transitions observed at higher LPDs suggest that these systems are more sensitive to changes in the London penetration constant, whereas the well-type behavior at the smallest LPD indicates a more complex and gradual response. These findings are crucial for understanding the stability and robustness of superconducting states in different materials and for optimizing superconducting devices for various applications.

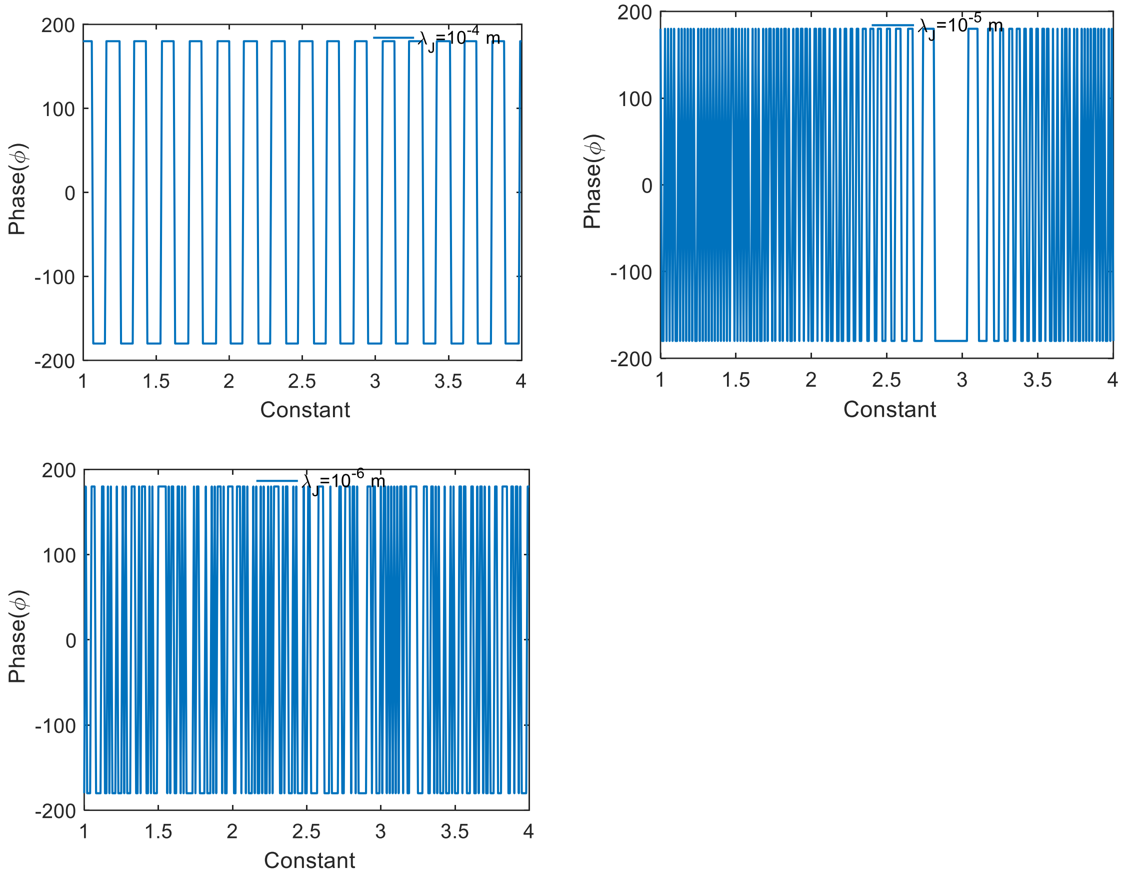

FIGURE 4. Phase with London Penetration Depth Constant at Different LPD

Perturbed Sine-Gordon Equation: Phase Dynamic in 3D Superconductor

FIGURE 4 presents the phase behavior in 3D superconductors as a function of the London penetration depth constant at various LPDs (10^{-4} m, 10^{-5} m, and 10^{-6} m) for perturbed sine-Gordon equation of 3D superconductors. The FIGURE 4 reveals distinct waveforms, specifically rectangular and square waveforms, corresponding to different LPDs, highlighting the complex interactions between the superconducting phase and the London penetration depth constant.

At an LPD of 10^{-4} m, the phase exhibits a square waveform. This square wave pattern suggests that the phase undergoes abrupt transitions between two states, likely corresponding to the alternating regions of high and low phase stability. The square waveform indicates a robust and well-defined phase transition, where the superconducting state switches sharply between different levels of coherence. The consistent square waveform at this LPD suggests a strong correlation between the superconducting phase and the London penetration depth constant, with the phase responding predictably to changes in the constant.

For LPDs of 10^{-5} m and 10^{-6} m, the phase behavior shifts to a rectangular waveform. The rectangular waveforms indicate that the phase transitions between two states are less abrupt compared to the square wave pattern observed at 10^{-4} m. The rectangular wave suggests a more gradual transition, where the phase spends more time in one state before shifting to another.

This behavior could be due to the reduced penetration depth, which allows for more nuanced and less abrupt interactions between the superconducting state and the external magnetic field. The rectangular waveform reflects the sensitivity of the superconducting phase to the London penetration depth constant, with the phase showing a more complex and less predictable response.

Interestingly, the oscillation of the rectangular wave before and after 3 Å of the London penetration depth constant is higher as we move further away from this value, either to the left or right. This suggests that the superconducting phase is more sensitive to changes in the penetration depth constant when it is significantly different from 3 Å. Near 3 Å, the oscillations are lower, indicating a relative stabilization of the phase, possibly due to a resonance effect or an optimal balance between the penetration depth and the coherence length.

This behavior highlights the critical nature of the 3 Å penetration depth constant, where the phase exhibits a transition from stability to instability depending on how far the system is from this critical value.

At an LPD of 10^{-6} m, the phase behavior becomes more complex, showing a mixture of both square and rectangular wave oscillations. This non-uniform pattern suggests that the superconducting phase is highly sensitive to variations in the London penetration depth constant, with no clear preference for a particular waveform. The presence of both high and low oscillations indicates that the phase is fluctuating between different states of stability and instability, possibly due to competing effects within the superconductor.

This complex behavior may arise from the intricate balance between the London penetration depth and the coherence length, where slight variations can lead to significant changes in the phase dynamics. The findings across all LPDs illustrate the diverse and intricate nature of phase behavior in 3D superconductors for perturbed sine-Gordon equation of 3D superconductors. The observed waveforms—square, rectangular, and mixed—underscore the influence of the London penetration depth constant on the superconducting phase. The transitions between different waveforms and the varying oscillation intensities highlight the sensitivity of the phase to both the penetration depth and the coherence length, emphasizing the need for precise control over these parameters in the design and application of superconducting materials.

Phase Dynamic in 3D Superconductor with London Penetration Depth

The phase variation with LPD, as shown in FIGURE 5, reveals how changes in phase impact the nature of waveforms within the superconducting system governed by the perturbed sine-Gordon equation of 3D superconductors. Specifically, the analysis highlights the presence of both rectangular and triangular waveforms, which exhibit distinct oscillatory behaviors depending on the degree of phase change. These waveforms are crucial in understanding the dynamic interactions within the system.

When the phase is altered by 1 degree, the resulting waveforms demonstrate higher oscillations below 50 Å of LPD. This indicates that, at smaller phase change, the system experiences more intense oscillations, particularly at lower LPD values. The increased intensity of these oscillations suggests that the superconducting system is highly sensitive to small phase variations, leading to more pronounced interactions at this LPD.

In contrast, when the phase change is increased to 20 degrees, both rectangular and triangular waveforms are still observed. However, the behavior of these waveforms differs from the 1-degree phase change scenario. Specifically, the width of the waveforms below 50 Å of LPD is found to be higher compared to the narrower waveforms seen at the smaller phase change. This broader width suggests that at a larger phase change, the system's response is more spread out or distributed over a wider range, indicating a less intense but more extensive interaction within the superconducting system.

The comparison between the phase changes of 1 degree and 20 degrees underscores the system's sensitivity to phase variations. The findings show that small phase changes result in more intense oscillations, while larger phase changes produce broader, less intense waveforms. These observations are significant for understanding the behavior of superconducting systems, particularly in low-dimensional environments where phase sensitivity plays a critical role.

The phase variation with LPD provides valuable insights into the superconducting phenomena. The relationship between LPD, phase changes, and the nature of the resulting waveforms can inform the fine-tuning and control of superconducting properties in various applications, especially in systems where maintaining specific superconducting parameters is essential.

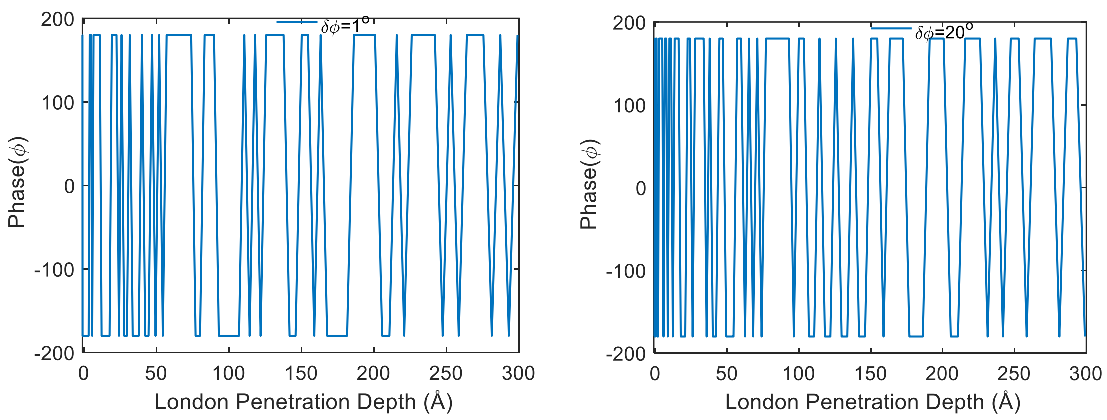

FIGURE 5. Phase Change with LPD at Different (1 and 20 degree) Phase Change

FIGURE 5 depicts a mixture of rectangular and triangular waves, similar to the behaviour described by Miller and Villagran [20]. At 1° phase change, intense oscillations occur, while a 20° change leads to broader, less intense waves. This finding highlights how phase sensitivity affects superconducting properties. This behavior is crucial for designing superconducting circuits where precise phase control is needed, especially in high-frequency applications.

In FIGURE 6, the phase variation with time for phase changes of 1 degree and 20 degrees is depicted, revealing a distinctive pattern resembling a Gaussian peak perturbed sine-Gordon equation of 3D superconductors. This pattern indicates that the phase initially starts low, increases to a maximum, and then decreases after reaching a peak. The repetition of this pattern forms what is known as multi-Gaussian peaks, characterized by multiple rises and falls in the phase over time.

For a phase change of 1 degree, the highest Gaussian peak occurs at approximately 0.55 ns. This suggests that the system reaches its maximum phase relatively quickly, with the phase rising and falling sharply around this point in time. The quick appearance of the peak and its relatively low position on the timeline indicate that the phase change is rapidly established and then begins to diminish, reflecting a transient behavior typical of small phase changes.

In contrast, when the phase change is increased to 20 degrees, the highest Gaussian peak is observed much later, at around 1.87 ns. This delay indicates that the system takes more time to reach its maximum phase, reflecting a slower, more gradual process compared to the 1-degree phase change. The extended time to reach the peak suggests that the system undergoes more complex interactions as the phase change is larger, requiring additional time for these interactions to fully develop.

Moreover, the occurrence of multi-Gaussian peaks is more pronounced with the 20-degree phase change, showing an increase in both the number and height of these peaks over time. This indicates that the phase continues to rise and fall in a more sustained manner as time progresses, reflecting the ongoing complexity and dynamics within the system. This behavior was not observed in the 1-degree phase change, where the phase quickly peaked and then subsided.

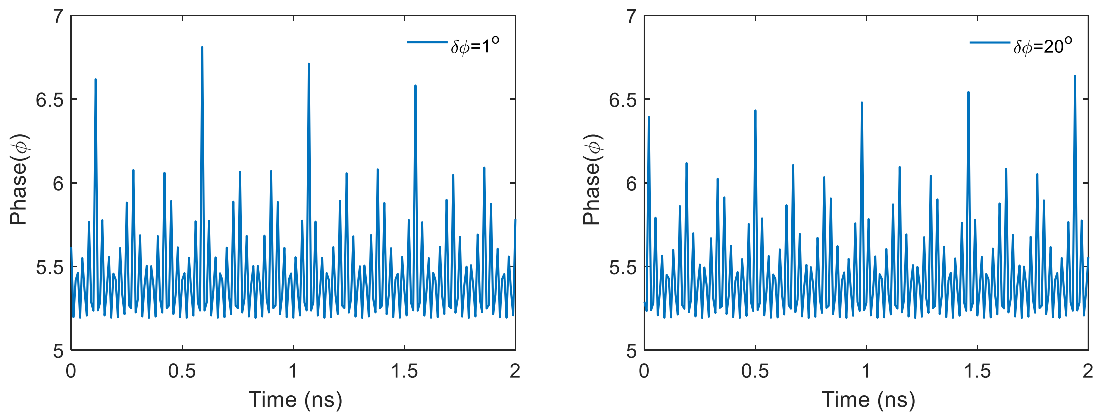

FIGURE 6. Phase Change with Time at Different (1 and 20 degree) Phase Change

The reasons behind these differences lie in the nature of the phase change itself. A smaller phase change, like 1 degree, leads to a quicker but less complex interaction, resulting in an early peak and a rapid decline in phase. In contrast, a larger phase change, such as 20 degrees, involves more substantial alterations within the system, leading to prolonged interactions and the emergence of multiple, more significant peaks over time. This reflects the increased energy and complexity associated with larger phase changes, which take longer to manifest fully within the system.

The detailed analysis of FIGURE 6 highlights the impact of phase change magnitude on the temporal evolution of phase variation, with significant differences in peak timing and the presence of multi-Gaussian peaks depending on the degree of phase change. The phase nature was found in FIGURE 6 is quite similar to the findings reported by Manzoor et al. [21].

FIGURE 6 depicts Gaussian peak patterns for phase changes of 1° and 20°. Smaller changes lead to quick peaks, while larger changes result in delayed, multi-Gaussian peaks due to complex energy interactions. This dynamic behavior is crucial for superconductors exposed to varying electromagnetic conditions, informing better designs for rapid-response superconducting systems.

These results provide a deeper understanding of how LPD and coherence length affect superconducting phases, expanding on previous studies. Insights gained here can guide the design of superconducting materials for improved stability and performance in devices. Future work can explore the influence of temperature and magnetic fields on phase behavior.

CONCLUSION

This study has provided novel insights into the phase dynamics of both low-dimensional and 3D superconductors, particularly in relation to the London penetration depth (LPD) and coherence length (ξ). The findings demonstrate that changes in coherence length and nanoparticle spacing significantly influence superconducting properties, such as magnetic field penetration and phase stability.

Specifically, in low-dimensional systems, we observed a complex, non-linear relationship between LPD and coherence length, driven by quantum confinement effects. In 3D superconductors, different phase shifts led to distinct waveform patterns, with smaller shifts resulting in higher oscillations and larger shifts producing broader, less intense waveforms.

These insights are crucial for optimizing superconducting devices. For instance, tuning LPD and coherence length can enhance the performance of superconducting circuits, qubits, and high-frequency applications where stable phase coherence is essential. Additionally, the findings on phase dynamics can inform the design of nanoparticle-based superconducting films to achieve better magnetic shielding and reduced energy dissipation.

For future research, it would be valuable to explore other superconducting materials or higher-dimensional models to validate and extend these findings. Investigating the impact of external factors, such as temperature fluctuations and magnetic field variations, on phase stability could further optimize the design of robust superconducting devices. These directions will help advance the practical applications of superconductors in emerging technologies, such as quantum computing, precision sensors, and efficient power transmission systems.

By elucidating the interplay between LPD, coherence length, and phase dynamics, this study lays the groundwork for future innovations in superconducting materials, providing a foundation for optimizing their performance in real-world applications.

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